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MOMENTS OF THE LOCATION OF THE MAXIMUM OF BROWNIAN MOTION WITH PARABOLIC DRIFT

SVANTE JANSON

ABSTRACT. We derive integral formulas, involving the Airy function, for moments of the time a two-sided Brownian motion with parabolic drift attains its maximum.

1. INTRODUCTION

Let $W(t)$ be a two-sided Brownian motion with $W(0) = 0$; i.e., $(W(t))_{t\geq 0}$ and $(W(-t))_{t\geq 0}$ are two independent standard Brownian motions. Fix $\gamma > 0$, and consider the Brownian motion with parabolic drift

$$W_\gamma(t) := W(t) - \gamma t^2. \tag{1.1}$$

We are interested in the maximum

$$M_\gamma := \max_{-\infty<t<\infty} W_\gamma(t) \tag{1.2}$$

of W_γ (which is a.s. finite), and, in particular, the *location* of the maximum, which we denote by

$$V_\gamma := \operatorname{argmax}_t\bigl(W_\gamma(t)\bigr); \tag{1.3}$$

in other words, $V_\gamma = t \iff W_\gamma(t) = M_\gamma$. (The maximum in (1.2) is a.s. attained at a unique point, so V_γ is well-defined a.s.)

The parameter γ is just a scale parameter, see Section 2, so it can be fixed arbitrarily without loss of generality.

The distribution of V_γ was called *Chernoff's distribution* by Groeneboom and Wellner [14] since it apparently first appeared in Chernoff [6]. It has been studied by several authors; in particular, Groeneboom [10, 11] gave a description of the distribution and Groeneboom and Wellner [14] give more explicit analytical and numerical formulas; see also Daniels and Skyrme [9]. It has many applications in statistics, see for example Groeneboom and Wellner [14] and the references given there, or, for a more recent example, Anevski and Soulier [3].

The purpose of the present paper is to give formulas for the moments of V_γ in terms of integrals involving the Airy function $\mathrm{Ai}(x)$. (Recall that $\mathrm{Ai}(x)$ satisfies $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and is up to a constant factor the unique solution that tends to 0 as $x \nearrow +\infty$. See further [1, 10.4].)

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All odd moments of V_γ vanish by symmetry, and our main result is the following formula for the even moments, proved in Section 5. (The special case $n=2$ is given by Groeneboom [13].)

Theorem 1.1. *For every even positive integer n, there is a polynomial p_n of degree at most $n/2$ such that*

$$\mathbb{E}\, V_\gamma^n = \frac{2^{-n/3}\gamma^{-2n/3}}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{p_n(z)}{\mathrm{Ai}(z)^2}\,\mathrm{d}z = \frac{2^{-n/3}\gamma^{-2n/3}}{2\pi} \int_{-\infty}^{\infty} \frac{p_n(\mathrm{i}y)}{\mathrm{Ai}(\mathrm{i}y)^2}\,\mathrm{d}y. \tag{1.4}$$

In particular, the variance of V_γ is

$$\mathbb{E}\, V_\gamma^2 = -\frac{2^{-2/3}\gamma^{-4/3}}{6\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{z}{\mathrm{Ai}(z)^2}\,\mathrm{d}z = \frac{2^{-2/3}\gamma^{-4/3}}{6\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{y}{\mathrm{Ai}(\mathrm{i}y)^2}\,\mathrm{d}y. \tag{1.5}$$

The integrals are rapidly converging and easily computed numerically by standard software.

The polynomials $p_n(z)$ can be found explicitly for any given n by the procedure in Section 5, but we do not know any general formula. They are given for small n in Table 1. See further the conjectures and problems in Section 6.

Numerical values of the first ten absolute moments are given by Groeneboom and Wellner [14]; the first four were computed by Groeneboom and Sommeijer (1984, unpublished). (Their values for the even moments agree with our formula.)

The maximum M_γ also appears in many applications. Its distribution is given in Groeneboom [10, 11, 12] and Daniels and Skyrme [9], see also, for example, Barbour [4, 5], Daniels [7, 8], Janson, Louchard and Martin-Löf [15]. (Groeneboom [11] describes even the joint distribution of the maximum M_γ and its location V_γ, see also Daniels and Skyrme [9].) Formulas for the mean are given by Daniels and Skyrme [9], see also Janson, Louchard and Martin-Löf [15]; in particular

$$\mathbb{E}\, M_\gamma = -\frac{2^{-2/3}\gamma^{-1/3}}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{z}{\mathrm{Ai}(z)^2}\,\mathrm{d}z. \tag{1.6}$$

Comparing (1.6) and (1.5), we find the simple relation [13]

$$\mathbb{E}\, V_\gamma^2 = \frac{1}{3\gamma}\,\mathbb{E}\, M_\gamma. \tag{1.7}$$

Since $M_\gamma = W_{V_\gamma} - \gamma V_\gamma^2$, this implies that, at the maximum point,

$$\mathbb{E}\, W_{V_\gamma} = \frac{4}{3}\,\mathbb{E}\, M_\gamma \qquad \text{and} \qquad \mathbb{E}\, \gamma V_\gamma^2 = \frac{1}{3}\,\mathbb{E}\, M_\gamma = \frac{1}{4}\,\mathbb{E}\, W_{V_\gamma}. \tag{1.8}$$

It would be interesting to find a direct proof of these simple relations.

Formulas for second and higher moments of M_γ are given in Janson, Louchard and Martin-Löf [15]; however, they are more complicated and do not correspond to the formulas for moments of V_γ in the present paper. It would be interesting to find relations between higher moments of M_γ and moments of V_γ.

Let us finally mention that the random variable V_γ is the value at a fixed time (0, to be precise) of the stationary stochastic process

$$V_\gamma(x) := \operatorname{argmax}_t\bigl(W(t) - \gamma(t-x)^2\bigr), \tag{1.9}$$

which is studied by Groeneboom [10, 11, 13]. It would be interesting to find formulas for joint moments of $V_\gamma(x)$, in particular for the covariances.

2. Scaling

For any $a > 0$, $W(at) \overset{\mathrm{d}}{=} a^{1/2}W(t)$ (as processes on $(-\infty,\infty)$), and thus

$$\begin{aligned} V_\gamma &= a \operatorname{argmax}\bigl(W(at) - \gamma(at)^2\bigr) \\ &\overset{\mathrm{d}}{=} a \operatorname{argmax}\bigl(a^{1/2}W(t) - a^2\gamma t^2\bigr) = aV_{a^{3/2}\gamma}. \end{aligned} \tag{2.1}$$

The parameter γ is thus just a scale parameter, and it suffices to consider a single choice of γ. Although the choices $\gamma = 1$ and $\gamma = 1/2$ seem most natural, we will use $\gamma = 1/\sqrt{2}$ which gives simpler formulas. We thus define $V := V_{1/\sqrt{2}}$ and have, for any γ,

$$V_\gamma \overset{\mathrm{d}}{=} 2^{-1/3}\gamma^{-2/3}V. \tag{2.2}$$

Similarly,

$$M_\gamma \overset{\mathrm{d}}{=} \gamma_1^{1/3}\gamma^{-1/3}M_{\gamma_1} \tag{2.3}$$

for any $\gamma, \gamma_1 > 0$.

3. First formulas for moments

By Groeneboom [11, Corollary 3.3], V has the density

$$f(x) = \tfrac{1}{2}g(x)g(-x), \tag{3.1}$$

where g has the Fourier transform, see [11, (3.8)] (this is where our choice $\gamma = 2^{-1/2}$ is convenient),

$$\widehat{g}(t) := \int_{-\infty}^{\infty} e^{\mathrm{i}tx} g(x)\,\mathrm{d}x = \frac{2^{1/2}}{\operatorname{Ai}(\mathrm{i}t)}, \qquad -\infty < t < \infty. \tag{3.2}$$

Note that $|\operatorname{Ai}(\mathrm{i}t)| \to \infty$ rapidly as $t \to \pm\infty$, see [1, 10.4.59] or [15, (A.3)], so $\widehat{g}(t)$ is rapidly decreasing. In fact, it follows from the precise asymptotic formula [1, 10.4.59] that

$$\operatorname{Ai}(x + \mathrm{i}y)^{-1} = O\bigl(e^{-cy^{3/2}}\bigr) \tag{3.3}$$

for some $c > 0$, uniformly for $|x| \le A$ for any fixed A and $|y| \ge 1$, say (to avoid the zeros of Ai). By differentiation (Cauchy's estimates, see e.g. [16, Theorem 10.25]), it follows that the same holds for all derivatives of $\operatorname{Ai}(x+\mathrm{i}y)^{-1}$, and thus all derivatives of $\widehat{g}(t)$ decrease rapidly. In particular, $\widehat{g}$ belongs to the Schwartz class $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}$; since this class is preserved by the Fourier transform, also $g \in \mathcal{S}$ (see e.g. [17, Theorem 7.7]). In particular, g is integrable.

The characteristic function of V is the Fourier transform $\widehat{f}(t)$, and thus by (3.1), (3.2) and standard Fourier analysis (see e.g. [17, Theorems 7.7–7.8], but note the different normalization chosen there), with $\check{g}(t) := g(-t)$,

$$\begin{aligned}\varphi(t) = \widehat{f}(t) &= \frac{1}{2}\widehat{g\check{g}}(t) = \frac{1}{2}\cdot\frac{1}{2\pi}\big(\widehat{g} * \widehat{\check{g}}\big) = \frac{1}{4\pi}\widehat{g} * \check{\widehat{g}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{\mathrm{Ai}(\mathrm{i}(t+s))\mathrm{Ai}(\mathrm{i}s)} = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}z}{\mathrm{Ai}(z+\mathrm{i}t)\mathrm{Ai}(z)}. \qquad (3.4)\end{aligned}$$

This is also given in Groeneboom [13, Lemma 2.1] (although with typos in the formula). The last integral is taken along the imaginary axis, but we can change the path of integration, as long as it passes to the right of the zeros a_n of the Airy function (which are real and negative), for example a line $\operatorname{Re} z = b$ with $b > a_1 = -|a_1|$.

We pause to note the following.

Theorem 3.1. *The moment generating function $\mathbb{E}\, e^{tV}$ is an entire function of t, and is given by, for any complex t,*

$$\mathbb{E}\, e^{tV} = \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty}\frac{\mathrm{d}z}{\mathrm{Ai}(z+t)\mathrm{Ai}(z)}, \qquad (3.5)$$

for any real b with $b > a_1$ and $b + \operatorname{Re} t > a_1$.

Proof. The density function $f(t) = \exp(-t^3/3)$ as $t \to \pm\infty$ by [11, Corollary 3.4(iii)]. Hence, $\mathbb{E}\, e^{tV} < \infty$ for every real t, which implies that $\mathbb{E}\, e^{tV}$ is an entire function of t. The formula (3.5) now follows from (3.4) by analytic continuation (for each fixed b). □

By differentiation of (3.4) (or (3.5)) we obtain, for any $n \geq 0$,

$$\mathbb{E}\, V^n = (-\mathrm{i})^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\varphi(0) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{1}{\mathrm{Ai}(z)}\right)\frac{\mathrm{d}z}{\mathrm{Ai}(z)}. \qquad (3.6)$$

By integration by parts, this is generalized to:

Theorem 3.2. *For any $j, k \geq 0$,*

$$\mathbb{E}\, V^{j+k} = \frac{(-1)^j}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\left(\frac{1}{\mathrm{Ai}(z)}\right)\cdot\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{1}{\mathrm{Ai}(z)}\right)\mathrm{d}z. \qquad (3.7)$$

Proof. For $j = 0$, this is (3.6).

If we denote the integral on the right-hand side of (3.7) by $J(j,k)$, then, for $j, k \geq 0$, integration by parts yields $A(j,k) = -A(j-1,k+1)$, and the result follows by induction. □

Since $\mathbb{E}\, V^{j+k} = \mathbb{E}\, V^{k+j}$, we see again by symmetry that $\mathbb{E}\, V^n = 0$ when n is odd. For even n, it is natural to take $j = k = n/2$ in (3.7). For small n, this yields the following examples. First, $n = j = k = 0$ yields

$$1 = \mathbb{E}\, V^0 = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{1}{\mathrm{Ai}(z)^2}, \qquad (3.8)$$

as noted by Daniels and Skyrme [9]; this is easily verified directly, since $\pi\mathrm{Bi}(z)/\mathrm{Ai}(z)$ is a primitive function of $1/\mathrm{Ai}^2$, see [2].

Next, for $n=2$ and $n=4$ we get

$$\mathbb{E}\,V^2 = \frac{-1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\mathrm{Ai}(z)}\right)\right)^2\mathrm{d}z = \frac{-1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{Ai}'(z)^2}{\mathrm{Ai}(z)^4}\,\mathrm{d}z \tag{3.9}$$

and

$$\begin{aligned}\mathbb{E}\,V^4 &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{\mathrm{Ai}(z)}\right)\right)^2\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(-\frac{z}{\mathrm{Ai}(z)}+\frac{2\mathrm{Ai}'(z)^2}{\mathrm{Ai}(z)^3}\right)^2\mathrm{d}z. \end{aligned}\tag{3.10}$$

Remark 3.3. Since $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$, it follows by induction that the m:th derivative $\frac{\mathrm{d}^m}{\mathrm{d}z^m}\big(\mathrm{Ai}(z)^{-1}\big)$ can be expressed as a linear combination (with integer coefficients) of terms

$$\frac{z^j\,\mathrm{Ai}'(z)^k}{\mathrm{Ai}(z)^\ell} \tag{3.11}$$

with $j,k\ge 0$, $2j+k\le m$ and $\ell = k+1$.

4. Some combinatorics of Airy integrals

Inspired by Remark 3.3, we define in general, for any integers $j,k,\ell\ge 0$ with $\ell > k$,

$$I(j,k,\ell) := \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{z^j\,\mathrm{Ai}'(z)^k}{\mathrm{Ai}(z)^\ell}\,\mathrm{d}z. \tag{4.1}$$

(The integrand decreases rapidly as $z\to\pm\mathrm{i}\infty$ because $\ell>k$, by (3.3) and $\mathrm{Ai}'(z)/\mathrm{Ai}(z)\sim -z^{1/2}$ as $|z|\to\infty$ in any sector $|\arg z|\le\pi-\varepsilon$ with $\varepsilon>0$ [1, 10.4.59 and 10.4.61]; thus the integral is absolutely convergent.) Then, recalling $\mathrm{Ai}''(z)=z\mathrm{Ai}(z)$, for any $j,k\ge 0$ and $\ell>k$,

$$\begin{aligned}0 &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^j\,\mathrm{Ai}'(z)^k}{\mathrm{Ai}(z)^\ell}\,\mathrm{d}z\\ &= jI(j-1,k,\ell)+kI(j+1,k-1,\ell-1)-\ell I(j,k+1,\ell+1),\end{aligned}\tag{4.2}$$

where we for convenience define $I(j,k,\ell)=0$ for $j<0$ or $k<0$. Consequently, for $j,k\ge 0$ and $\ell>k$,

$$I(j,k+1,\ell+1) = \frac{j}{\ell}I(j-1,k,\ell)+\frac{k}{\ell}I(j+1,k-1,\ell-1), \tag{4.3}$$

or, for $j\ge 0$, and $\ell>k\ge 1$,

$$I(j,k,\ell) = \begin{cases}\frac{j}{\ell-1}I(j-1,0,\ell-1), & k=1,\\ \frac{j}{\ell-1}I(j-1,k-1,\ell-1)+\frac{k-1}{\ell-1}I(j+1,k-2,\ell-2), & k\ge 2.\end{cases} \tag{4.4}$$

By repeatedly using this relation, any $I(j,k,\ell)$ may be expressed as a rational combination of $I(p,0,\ell-k)$ with $0\le p\le j+k/2$. For example,

$$I(0,1,3)=0; \tag{4.5}$$
$$I(0,2,4)=\tfrac13 I(1,0,2); \tag{4.6}$$
$$I(1,2,4)=\tfrac13 I(0,1,3)+\tfrac13 I(2,0,2)=\tfrac13 I(2,0,2); \tag{4.7}$$
$$I(0,4,6)=\tfrac35 I(1,2,4)=\tfrac15 I(2,0,2). \tag{4.8}$$

5. Back to moments

Proof of Theorem 1.1. By combining (3.6) and Remark 3.3, we can express any moment $\mathbb{E}\,V^n$ as a linear combination with integer coefficients of terms $I(j,k,k+2)$ with $2j+k\le n$. By repeated use of (4.4) (see the comment after it), this can be further developed into a linear combination with rational coefficients of terms $I(j,0,2)$ with $0\le j\le n/2$.

For example, by (3.9) and (4.6),

$$\mathbb{E}\,V^2=-I(0,2,4)=-\frac13 I(1,0,2)=-\frac{1}{6\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{z}{\operatorname{Ai}(z)^2}\,\mathrm{d}z, \tag{5.1}$$

which yields (1.5) by (2.2).

To continue with higher moments we have next, by (3.10) and (4.7)–(4.8),

$$\begin{aligned}\mathbb{E}\,V^4&=I(2,0,2)-4I(1,2,4)+4I(0,4,6)\\&=\Bigl(1-\frac43+\frac45\Bigr)I(2,0,2)=\frac{7}{15}I(2,0,2)\\&=\frac{7}{30\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{z^2}{\operatorname{Ai}(z)^2}\,\mathrm{d}z.\end{aligned} \tag{5.2}$$

Similarly (using Maple),

$$\mathbb{E}\,V^6=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{(26-31z^3)/21}{\operatorname{Ai}(z)^2}\,\mathrm{d}z. \tag{5.3}$$

In general this procedure yields, for some rational numbers b_{nj},

$$\mathbb{E}\,V^n=\sum_{j=0}^{n/2}b_{nj}I(j,0,2)=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{p_n(z)}{\operatorname{Ai}(z)^2}\,\mathrm{d}z. \tag{5.4}$$

where $p_n(z):=\sum_{j=0}^{n/2}b_{nj}z^j$ is a polynomial of degree at most $n/2$. By (2.2), this is equivalent to the more general (1.4). $\square$

For odd n, we already know that $\mathbb{E}\,V^n=0$, so we are mainly interested in p_n for even n. The polynomials p_0,p_2,p_4,p_6 are implicit in (3.8), (5.1), (5.2) and (5.3), and some further cases (computed with Maple) are given in Table 1.

$$
\begin{aligned}
p_0(z) &= 1\\
p_2(z) &= -\tfrac{1}{3}z\\
p_4(z) &= \tfrac{7}{15}z^2\\
p_6(z) &= -\tfrac{31}{21}z^3 + \tfrac{26}{21}\\
p_8(z) &= \tfrac{127}{15}z^4 - \tfrac{196}{9}z\\
p_{10}(z) &= -\tfrac{2555}{33}z^5 + \tfrac{13160}{33}z^2\\
p_{12}(z) &= \tfrac{1414477}{1365}z^6 - \tfrac{2419532}{273}z^3 + \tfrac{1989472}{1365}.
\end{aligned}
$$

TABLE 1. The polynomials $p_n(z)$ for small even n.

Remark 5.1. We can see from Table 1 that (for these n) $p_n(z)$ contains only terms z^j where $j \equiv n/2 \pmod 3$. This is easily verified for all even n: a closer look at the induction in Remark 3.3 shows that only terms (3.11) with $2j+k \equiv m \pmod 3$ appears, and the reduction in (4.4) preserves $2j+k \pmod 3$.

6. Problems and conjectures

The proof above yields an algorithm for computing the polynomials $p_n(z)$, but no simple formula for them. We thus ask the following.

Problem 6.1. Is there an explicit formula for the coefficients b_{nj}, and thus for the polynomials $p_n(z)$? Perhaps a recursion formula?

We have computed $p_n(z)$ for $1 \le n \le 100$ by Maple, and based on the results (see also Table 1), we make the following conjectures.

Conjectures 6.2. (i) $p_n(z) = 0$ *for every odd* n.

(ii) $p_n(z)$ *has degree exactly* $n/2$*; i.e., the coefficient* $b_{n,n/2}$ *of* $z^{n/2}$ *is non-zero.*

(iii) *These leading coefficients have exponential generating function*

$$\sum_{n=0}^{\infty} \frac{b_{n,n/2}}{n!} x^n = \frac{x}{\sinh x}. \tag{6.1}$$

Of these conjectures, (i) is natural, since we know that $\mathbb{E}\, V^n = 0$ for odd n, and (ii) is not surprising. The precise conjecture (6.1) is perhaps more surprising. We have verified that the coefficients up to x^{100} agree, but we have no general proof.

The simple form of (6.1) suggests also the following open problem.

Problem 6.3. Is there an explicit formula for the generating function $\sum_{n=0}^{\infty} p_n(z)x^n$?

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Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`

URL: `http://www.math.uu.se/~svante/`